Matter-field theory of the Casimir force *

Masato Koashi[1] and Masahito Ueda[2]
[1] NTT Basic Research Laboratories, 3-1 Morinosato Wakamiya, Atsugi, Kanagawa 243-0198, Japan
[2] Department of Physical Electronics, Hiroshima University, Higashi-Hiroshima 739-8527, Japan

A matter-field theory of the Casimir force is formulated in which the electromagnetic field and collective modes of dielectric media are treated on an equal footing. In our theory, the Casimir force is attributed to zero-point energies of the combined matter-field modes. We analyze why some of the existing theories favor the interpretation of the Casimir force as originating from zero-point energies of the electromagnetic field and others from those of the matter.

PACS numbers: 12.20.Ds, 42.50.Ct, 42.50.Lc

I. INTRODUCTION

It is well known that two conducting neutral plates placed at a small distance attract each other. This phenomenon is often called the Casimir effect because of the celebrated formula he derived for the force between two perfectly conducting plates [1]. Casimir calculated the sum of the quantum-mechanical zero-point energies of the normal modes of the electromagnetic (EM) field between the two metal plates, and showed that the total energy depends on the distance between the plates. The spatial derivative of the sum gives what we now call the Casimir force. The generalization of the Casimir effect to the case of two dielectric plates was made by Lifshitz [2], and his formula was rederived by van Kampen, Nijboer, and Schram [3,4] using a technique attributed to Casimir. The Casimir force is essentially a long-range van der Waals force, where one cannot ignore the delay caused by the finiteness of the velocity of light. This retardation effect is not always a small correction and may alter the very nature of the force. For example, the Casimir force between two perfectly conducting hemispherical shells has been shown to be repulsive [5].

Experimental studies of the Casimir force started long ago [6,7,8] and have culminated in a recent high-resolution measurement by Lamoreaux [9]. A closely related phenomenon, the retarded attraction force between an atom and a metal plate (the Casimir-Polder force [10]), was also measured successfully by Sukenik *et al.* [11]. With further improvements, it will hopefully be possible to confirm experimentally the theoretical predictions of several corrections, such as finite-temperature effects [12,13,14] and radiative corrections [15].

The scope of the research of the Casimir effect covers many areas of physics and other fields, ranging from biology to cosmology and elementary particle physics. Reviews and extensive references are available, for example, in Refs. [16,17,18,19].

The Casimir effect has usually been attributed to zero-point fluctuations of the EM field. However, Schwinger and his collaborators [14,20] showed that the Casimir effect can be derived in terms of Schwinger's source theory, which has no explicit reference to vacuum-field fluctuations of the EM field. Milonni and Shih have recently developed a source theory of the Casimir effect within the framework of conventional quantum electrodynamics [21]. In this theory, the Casimir force originates from quantum fluctuations of atomic dipoles in the dielectric, and the EM field plays only the passive role of mediating interactions between those dipoles.

It seems like it is only a matter of taste whether we attribute the Casimir effect to the quantum nature of the EM field or to that of the matter. Accordingly, the following questions naturally arise: Do the field and the matter really play (or to what extent do they play) symmetrical roles in this problem? Why do some approaches emphasize the quantum nature of the EM field and others stress that of the matter? To answer these questions, it would be of interest to discuss this problem from a standpoint that has no preference for the field or for the matter. Unfortunately, existing theories do not suit this purpose because they all invoke in one way or another the Maxwell equations in which the degrees of freedom of the matter are in advance embedded in the frequency dependence of the dielectric response function.

The primary purpose of this paper is to propose a matter-field theory of the Casimir force in which the matter and the field are treated on an equal footing. Our strategy is to explicitly diagonalize the matter-field Hamiltonian which is quadratic in its dynamical variables. Because relevant physical quantities are then expressed in terms of eigenvalues and eigenvectors of the full Hamiltonian, all physical effects allow unambiguous interpretation.

---

*To be published in Phys. Rev. A

This paper is organized as follows. Section II shows how the Casimir force is derived, starting from the Lagrangian describing the interaction between the EM field and collective modes of the matter. Section III proves that this derivation leads to the same results as existing theories [1,2,3,4,14,20,21]. It will be shown that in our treatment both field and matter contributes zero-point energies. Section IV analyzes the previous derivations of the Casimir force and points out that an inherent asymmetry indeed exists in the problem between the field and the matter, which favors the interpretation of the Casimir force as originating from zero-point fluctuations of the EM field in some formulations [3,4], and from those of the matter in others [21]. Section V summarizes the main conclusions of this paper and describes our answers to the above questions.

II. MATTER-FIELD THEORY

A. Formulation of the problem

Consider a system in which two dielectric slabs are separated by the vacuum of the EM field as shown in Fig. 1. We assume that the entire system V is enclosed by a perfectly conducting metal, and denote the boundary of the system on the metal as S. The region V_m occupied by the dielectric is surrounded by S and by the interface S_m between the dielectric and the vacuum. The following discussion does not depend on concrete shapes of V and V_m, but we require that any closed loop or any closed surface in V can continuously shrink to a point without going outside of V. The condition concerning closed loops ensures that any irrotational vector function $\mathbf{Q}(\mathbf{r})$ [i.e., $\boldsymbol{\nabla} \times \mathbf{Q}(\mathbf{r}) = \mathbf{0}$] has a scalar potential $\phi(\mathbf{r})$ with $-\boldsymbol{\nabla}\phi(\mathbf{r}) = \mathbf{Q}(\mathbf{r})$. The condition concerning closed surfaces ensures that if $\mathbf{Q}(\mathbf{r})$ has a scalar potential and $\mathbf{Q} \perp S$, $\mathbf{Q}(\mathbf{r})$ has a scalar potential ϕ that vanishes everywhere on S. We impose the same requirements on the topology of V_m.

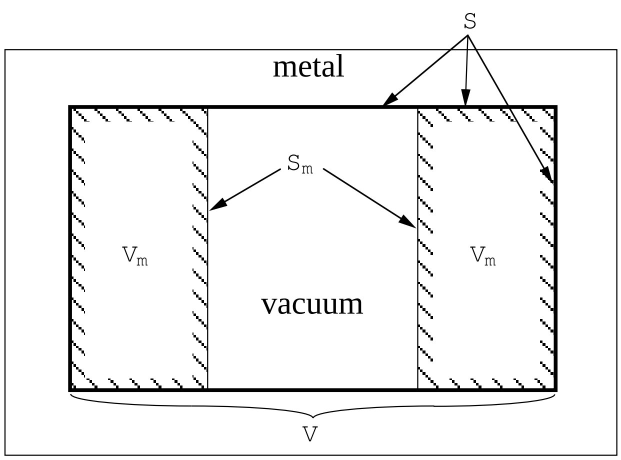

FIG. 1. Two dielectric slabs (region V_m) are separated by the vacuum. The entire system V is enclosed by a perfectly conducting metal. The boundary of the system on the metal is denoted by S, and the boundary between the dielectric slabs and the vacuum of the EM field is denoted by S_m.

In a field theory of the Casimir force [4], normal-mode frequencies of the "EM field" are determined from the Maxwell equations with proper boundary conditions, and the degrees of freedom of the dielectric slabs do not appear explicitly because the effect of the presence of the dielectric is incorporated into the theory through the dielectric response function $\epsilon(\omega)$. In a matter theory of the Casimir force [21], the Green function of the EM field, which mediates the interaction between atomic dipoles, is determined from the same dielectric response function. Because our goal is to treat the EM field and the matter on an equal footing, we first consider the EM field and the dielectric media separately, and then introduce the interaction between them.

The Lagrangian L_{field} of the free EM field is written as

$$L_{\text{field}} = \int_V dv \left(\frac{\epsilon_0}{2} |\mathbf{E}|^2 - \frac{1}{2\mu_0} |\mathbf{B}|^2 \right), \tag{1}$$

where we assume that the EM field is only subject to a perfectly conducting boundary S.

We take the following simple model for the dielectric: It consists of various kinds of particles whose species are labeled by j with number density n_j. The jth species has mass m_j and charge e_j, and particles belonging to this

species are bounded around their equilibrium positions by a restoring force characterized by frequency ω_j. In order to describe their collective motions, we use a complete orthonormal set $\{\boldsymbol{\beta}_i(\mathbf{r})\}$ of mode functions in the region V_m that is occupied by the dielectric. They satisfy

$$\int_{V_m} \boldsymbol{\beta}_i(\mathbf{r}) \cdot \boldsymbol{\beta}_j(\mathbf{r}) dv = \delta_{ij}, \tag{2}$$

where δ_{ij} is the Kronecker's delta. We take the boundary condition on S as $\boldsymbol{\beta}_i \parallel S$ and that on S_m as $\boldsymbol{\beta}_i \parallel S_m$. Note that imposing a boundary condition does not alter the physics in our model because we assume no interaction (except that mediated by the EM field) between the particles. Under this boundary condition, the set $\{\boldsymbol{\beta}_i\}$ is uniquely classified into the set of transverse vector fields $\{\boldsymbol{\beta}_i^{(T)}\}$ satisfying $\boldsymbol{\nabla} \cdot \boldsymbol{\beta}_i^{(T)} = 0$, and the set of longitudinal ones $\{\boldsymbol{\beta}_i^{(L)}\}$ satisfying $\boldsymbol{\nabla} \times \boldsymbol{\beta}_i^{(L)} = \mathbf{0}$.

The polarization density $\mathbf{P}_j(\mathbf{r})$ of the jth species can be expanded in terms of these bases as

$$\mathbf{P}_j(\mathbf{r}) = \sqrt{\epsilon_0} \sum_i \boldsymbol{\beta}_i^{(T)}(\mathbf{r}) X_{ij}^{(T)} + \sqrt{\epsilon_0} \sum_i \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) X_{ij}^{(L)}, \tag{3}$$

where $X_{ij}^{(T)}$ and $X_{ij}^{(L)}$ are the generalized coordinates that describe the collective displacements of the jth species. The corresponding current density $\mathbf{j}_j(\mathbf{r})$ and the charge density $\rho_j(\mathbf{r})$ are given by

$$\mathbf{j}_j(\mathbf{r}) = \dot{\mathbf{P}}_j(\mathbf{r}) = \sqrt{\epsilon_0} \sum_i \boldsymbol{\beta}_i^{(T)}(\mathbf{r}) \dot{X}_{ij}^{(T)} + \sqrt{\epsilon_0} \sum_i \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) \dot{X}_{ij}^{(L)} \tag{4}$$

and

$$\rho_j(\mathbf{r}) = -\boldsymbol{\nabla} \cdot \mathbf{P}_j(\mathbf{r}) = -\sqrt{\epsilon_0} \sum_i \boldsymbol{\nabla} \cdot \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) X_{ij}^{(L)}. \tag{5}$$

In Eq. (4), we have neglected nonlinear terms in $X_{ij}^{(T,L)}$, which give rise to the term $\mathbf{v} \times \mathbf{B}$ of the Lorentz force. They may provide a relativistic correction to the Casimir force, which, however, does not arise in the lowest nontrivial order we are concerned with. The total kinetic energy of the jth particle is written as follows:

$$T_j = \int_{V_m} dv \frac{1}{2} n_j m_j \left(\frac{\mathbf{j}_j(\mathbf{r})}{e_j n_j} \right)^2 = \sum_j \frac{1}{2\omega_{pj}^2} (\sum_i \dot{X}_{ij}^{(T)2} + \sum_i \dot{X}_{ij}^{(L)2}), \tag{6}$$

where $\omega_{pj} \equiv \sqrt{e_j^2 n_j / \epsilon_0 m_j}$ is the plasma frequency. The Lagrangian for the dielectric L_{matter} is thus given by

$$L_{\mathrm{matter}} = \sum_{ij} \frac{1}{2\omega_{pj}^2} (\dot{X}_{ij}^{(T)2} - \omega_{0j}^2 X_{ij}^{(T)2}) + \sum_{ij} \frac{1}{2\omega_{pj}^2} (\dot{X}_{ij}^{(L)2} - \omega_{0j}^2 X_{ij}^{(L)2}). \tag{7}$$

The total Lagrangian L for the whole system is the sum of L_{field}, L_{matter}, and the interaction part:

$$L = L_{\mathrm{field}} + L_{\mathrm{matter}} - \sum_j \int_{V_m} dv \left[\rho_j(\mathbf{r}) \varphi(\mathbf{r}) - \mathbf{j}_j(\mathbf{r}) \cdot \mathbf{A}(\mathbf{r}) \right], \tag{8}$$

where $\varphi(\mathbf{r})$ and $\mathbf{A}(\mathbf{r})$ are respectively the scalar and the vector potential of the EM field. The corresponding Hamiltonian in the Coulomb gauge is derived as follows:

$$\begin{aligned} H &= \sum_{ij} \left(\dot{X}_{ij}^{(T)} \frac{\partial L}{\partial \dot{X}_{ij}^{(T)}} + \dot{X}_{ij}^{(L)} \frac{\partial L}{\partial \dot{X}_{ij}^{(L)}} \right) + \dot{\mathbf{A}}(\mathbf{r}) \cdot \frac{\delta L}{\delta \dot{\mathbf{A}}(\mathbf{r})} - L \\ &= \sum_{ij} \left[\frac{\omega_{pj}^2}{2} (P_{ij}^{(T)} - \sqrt{\epsilon_0} \int_{V_m} \boldsymbol{\beta}_i^{(T)} \cdot \mathbf{A} dv)^2 + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(T)2} \right] \\ &\quad + \sum_{ij} \left[\frac{\omega_{pj}^2}{2} (P_{ij}^{(L)} - \sqrt{\epsilon_0} \int_{V_m} \boldsymbol{\beta}_i^{(L)} \cdot \mathbf{A} dv)^2 + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(L)2} \right] \\ &\quad + \int_V dv \left(\frac{\epsilon_0}{2} |\mathbf{E}^{(T)}|^2 + \frac{1}{2\mu_0} |\mathbf{B}|^2 \right) + \int_V \frac{\epsilon_0}{2} |\mathbf{E}^{(L)}|^2 dv, \end{aligned} \tag{9}$$

where $\mathbf{E}^{(T)}$ and $\mathbf{E}^{(L)}$ are respectively the transverse and the longitudinal parts of the electric field; $P_{ij}^{(T)} \equiv \partial L/\partial \dot{X}_{ij}^{(T)}$ and $P_{ij}^{(L)} \equiv \partial L/\partial \dot{X}_{ij}^{(L)}$ are the generalized momenta conjugate to $X_{ij}^{(T)}$ and $X_{ij}^{(L)}$, respectively.

The transverse part of the EM field can be decomposed into the normal modes in the usual manner. Consider a complete orthonormal set of transverse mode functions $\{\boldsymbol{\alpha}_i^{(T)}(\mathbf{r})\}$ in V that satisfy

$$\triangle^2 \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) = -\frac{\Omega_i^2}{c^2}\boldsymbol{\alpha}_i^{(T)}(\mathbf{r}), \tag{10}$$

$$\int_V \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_j^{(T)}(\mathbf{r}) dv = \delta_{ij}. \tag{11}$$

Because S is the boundary on the perfect conductor, $\boldsymbol{\alpha}_i^{(T)}$ should satisfy the boundary condition $\boldsymbol{\alpha}_i^{(T)} \perp S$. The transverse fields $\mathbf{A}(\mathbf{r})$ and $\mathbf{E}^{(T)}(\mathbf{r})$ can be expanded with this base as

$$\mathbf{A}(\mathbf{r}) = \frac{1}{\sqrt{\epsilon_0}} \sum_i \boldsymbol{\alpha}^{(T)}(\mathbf{r}) q_i \tag{12}$$

and

$$\mathbf{E}^{(T)}(\mathbf{r}) = -\frac{1}{\sqrt{\epsilon_0}} \sum_i \boldsymbol{\alpha}^{(T)}(\mathbf{r}) p_i, \tag{13}$$

where q_i and p_i are canonically conjugate pairs. The transverse EM field part of the Hamiltonian (9) then reduces to a collection of harmonic oscillators:

$$\int_V dv \left(\frac{\epsilon_0}{2}|\mathbf{E}^{(T)}|^2 + \frac{1}{2\mu_0}|\mathbf{B}|^2 \right) = \sum_i \left(\frac{1}{2}p_i^2 + \frac{1}{2}\Omega_i^2 q_i^2 \right). \tag{14}$$

In the Coulomb gauge, the longitudinal part $\mathbf{E}^{(L)}(\mathbf{r})$ of the electric field satisfies the Poisson equation $\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E}^{(L)} = \rho$, where the total charge ρ is given from Eq. (5) as

$$\rho = \sum_j \rho_j = -\sqrt{\epsilon_0} \sum_{ij} \boldsymbol{\nabla} \cdot \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) X_{ij}^{(L)} \tag{15}$$

in V_m and zero elsewhere. Let us define the longitudinal mode function $\boldsymbol{\alpha}_i^{(L)}(\mathbf{r})$ in V that satisfies the boundary condition $\boldsymbol{\alpha}_i^{(L)} \perp S$ and

$$\boldsymbol{\nabla} \cdot \boldsymbol{\alpha}_i^{(L)}(\mathbf{r}) = \begin{cases} \boldsymbol{\nabla} \cdot \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) & \text{in } V_m, \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

The solution to Eq. (16) is given as

$$\boldsymbol{\alpha}_i^{(L)}(\mathbf{r}) = \begin{cases} \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) - \sum_k \boldsymbol{\alpha}_k^{(T)}(\mathbf{r}) \int_{V_m} \boldsymbol{\alpha}_k^{(T)}(\mathbf{r}') \cdot \boldsymbol{\beta}_i^{(L)}(\mathbf{r}') dv' & \text{in } V_m, \\ - \sum_k \boldsymbol{\alpha}_k^{(T)}(\mathbf{r}) \int_{V_m} \boldsymbol{\alpha}_k^{(T)}(\mathbf{r}') \cdot \boldsymbol{\beta}_i^{(L)}(\mathbf{r}') dv' & \text{otherwise.} \end{cases} \tag{17}$$

Note that $\boldsymbol{\alpha}_i^{(L)}(\mathbf{r})$ is neither normalized nor orthogonal. With this base and $X_{ij}^{(L)}$, $\mathbf{E}^{(L)}(\mathbf{r})$ can now be expanded as follows:

$$\mathbf{E}^{(L)}(\mathbf{r}) = -\frac{1}{\sqrt{\epsilon_0}} \sum_{ij} \boldsymbol{\alpha}_i^{(L)}(\mathbf{r}) X_{ij}^{(L)}. \tag{18}$$

The Coulomb-energy part of the Hamiltonian (9) is

$$\int_V \frac{\epsilon_0}{2}|\mathbf{E}^{(L)}|^2 dv = \frac{1}{2} \sum_{ik} \left(\sum_j X_{ij}^{(L)} \right) \left(\sum_{j'} X_{kj'}^{(L)} \right) \int_V \boldsymbol{\alpha}_i^{(L)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv. \tag{19}$$

Using Eq. (17) and the orthogonality between $\boldsymbol{\alpha}_i^{(T)}(\mathbf{r})$ and $\boldsymbol{\alpha}_k^{(L)}(\mathbf{r})$, the integral on the right-hand side of Eq. (19) can be rewritten as follows:

$$\int_V \boldsymbol{\alpha}_i^{(L)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv = \int_{V_m} \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv. \tag{20}$$

Combining Eqs. (9), (12), (14), (19), and (20), we obtain the following expression for the total Hamiltonian:

$$\begin{aligned} H = &\sum_{ij} \left[\frac{\omega_{pj}^2}{2} \left(P_{ij}^{(T)} - \sum_k q_k \int_{V_m} \boldsymbol{\alpha}_k^{(T)} \cdot \boldsymbol{\beta}_i^{(T)} dv \right)^2 + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(T)2} \right] \\ &+ \sum_{ij} \left[\frac{\omega_{pj}^2}{2} \left(P_{ij}^{(L)} - \sum_k q_k \int_{V_m} \boldsymbol{\alpha}_k^{(T)} \cdot \boldsymbol{\beta}_i^{(L)} dv \right)^2 + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(L)2} \right] \\ &+ \sum_i \left(\frac{1}{2} p_i^2 + \frac{1}{2} \Omega_i^2 q_i^2 \right) \\ &+ \frac{1}{2} \sum_{ik} \left(\sum_j X_{ij}^{(L)} \right) \left(\sum_{j'} X_{kj'}^{(L)} \right) \int_{V_m} \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv. \end{aligned} \tag{21}$$

B. Plasmons

In this section, we discuss the physical implications of the longitudinal displacement represented by $\{\boldsymbol{\beta}_i^{(L)}(\mathbf{r})\}$. Let us consider another set of complete orthonormal modes $\{\tilde{\boldsymbol{\beta}}_i(\mathbf{r})\}$ in V_m that satisfy different boundary conditions on the boundaries, namely, $\tilde{\boldsymbol{\beta}}_i \perp S$ and $\tilde{\boldsymbol{\beta}}_i \perp S_m$. Under these new boundary conditions, the set $\{\tilde{\boldsymbol{\beta}}_i\}$ is uniquely separated into the set of transverse vector fields $\{\tilde{\boldsymbol{\beta}}_i^{(T)}\}$ satisfying $\boldsymbol{\nabla} \cdot \tilde{\boldsymbol{\beta}}_i^{(T)} = 0$, and the set of longitudinal ones $\{\tilde{\boldsymbol{\beta}}_i^{(L)}\}$, which have scalar potentials $\tilde{\phi}_i(\mathbf{r})$ ($\tilde{\boldsymbol{\beta}}_i^{(L)} = -\boldsymbol{\nabla}\tilde{\phi}_i$) that vanish on S and S_m.

The orthogonality between $\tilde{\boldsymbol{\beta}}_k^{(L)}$ and $\boldsymbol{\beta}_i^{(T)}$ is shown as follows:

$$\int_{V_m} \boldsymbol{\beta}_i^{(T)}(\mathbf{r}) \cdot \tilde{\boldsymbol{\beta}}_k^{(L)}(\mathbf{r}) dv = -\int \mathbf{n} \cdot \boldsymbol{\beta}_i^{(T)}(\mathbf{r}) \tilde{\phi}_k(\mathbf{r}) dS = 0, \tag{22}$$

where the surface integral runs over the surface of V_m, and $\mathbf{n}$ is the unit vector normal to that surface. In contrast, $\tilde{\boldsymbol{\beta}}_i^{(T)}$ and $\boldsymbol{\beta}_k^{(L)}$ are not necessarily orthogonal. This implies that a vector longitudinal under the boundary conditions $\boldsymbol{\beta}_i \parallel S, S_m$ is not necessarily longitudinal under the boundary conditions $\tilde{\boldsymbol{\beta}}_i \perp S, S_m$. Let us choose the base $\{\boldsymbol{\beta}_i^{(L)}\}$ as a union of two such orthogonal sets, $\{\boldsymbol{\beta}_i^{(b)}\}$ and $\{\boldsymbol{\beta}_i^{(s)}\}$, that $\boldsymbol{\beta}_i^{(b)}$ can be expanded with $\{\tilde{\boldsymbol{\beta}}_i^{(L)}\}$, and $\boldsymbol{\beta}_i^{(s)}$ with $\{\tilde{\boldsymbol{\beta}}_i^{(T)}\}$. Associated with $\boldsymbol{\beta}_i^{(b)}$ and $\boldsymbol{\beta}_i^{(s)}$, we may introduce $\boldsymbol{\alpha}_i^{(b)}$ and $\boldsymbol{\alpha}_i^{(s)}$ through the relation (17).

Since $\boldsymbol{\beta}_i^{(b)}$ has a scalar potential $\phi_i^{(b)}$ ($\boldsymbol{\beta}_i^{(b)} = -\boldsymbol{\nabla}\phi_i^{(b)}$) that vanishes on the boundaries S and S_m, part of the integral (20) that includes the $\boldsymbol{\beta}_i^{(b)}$ mode can be further simplified to be

$$\begin{aligned} \int_{V_m} \boldsymbol{\beta}_i^{(b)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv &= \int_{V_m} \phi_i^{(b)}(\mathbf{r}) \boldsymbol{\nabla} \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv = \int_{V_m} \phi_i^{(b)}(\mathbf{r}) \boldsymbol{\nabla} \cdot \boldsymbol{\beta}_k^{(L)}(\mathbf{r}) dv \\ &= \int_{V_m} \boldsymbol{\beta}_i^{(b)}(\mathbf{r}) \cdot \boldsymbol{\beta}_k^{(L)}(\mathbf{r}) dv, \end{aligned} \tag{23}$$

where we used Eq. (16) in deriving the second equality. Because of the orthogonality condition between $\boldsymbol{\beta}_i^{(b)}$ and $\boldsymbol{\beta}_k^{(s)}$, we obtain

$$\int_{V_m} \boldsymbol{\beta}_i^{(b)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) dv = 0 \tag{24}$$

and because of the orthonormality of $\{\boldsymbol{\beta}_i^{(b)}\}$, we obtain

$$\int_{V_m} \boldsymbol{\beta}_i^{(b)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(b)}(\mathbf{r}) dv = \delta_{ik}. \tag{25}$$

From these, the static Coulomb-interaction part of the Hamiltonian (21) can be decomposed into two parts:

$$\begin{aligned} &\frac{1}{2}\sum_{ik}\left(\sum_j X_{ij}^{(L)}\right)\left(\sum_{j'} X_{kj'}^{(L)}\right)\int_{V_m} \boldsymbol{\beta}_i^{(L)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(L)}(\mathbf{r}) dv \\ &= \frac{1}{2}\sum_{ik}\left(\sum_j X_{ij}^{(s)}\right)\left(\sum_{j'} X_{kj'}^{(s)}\right)\int_{V_m} \boldsymbol{\beta}_i^{(s)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) dv + \frac{1}{2}\sum_i\left(\sum_j X_{ij}^{(b)}\right)^2 . \end{aligned} \tag{26}$$

This result shows that the Coulomb interaction between the modes $X_{ij}^{(b)}$ does not depend on the boundaries (bulk plasmons). Further, these modes do not interact with the transverse EM field since

$$\int_{V_m} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \cdot \boldsymbol{\beta}_k^{(b)}(\mathbf{r}) dv = -\int \mathbf{n} \cdot \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \phi_k^{(b)}(\mathbf{r}) dS = 0, \tag{27}$$

where the surface integral runs over the surface of V_m. The bulk plasmons thus do not contribute to the Casimir force.

The modes $X_{ij}^{(s)}$ describe the surface plasmons whose resonance frequencies do depend on the parameters of the surface of V_m and its outside, through the integral in Eq. (26). The surface plasmons are also affected by the surroundings through the interaction with the transverse EM field, for their scalar potentials by definition should not vanish everywhere on the surface of V_m and the integral $\int_{V_m} \boldsymbol{\alpha}_i^{(T)} \boldsymbol{\beta}_k^{(s)} dv$ does not vanish unlike Eq. (27).

C. Combined oscillations

After separating the longitudinal modes into the two types of plasmons, the total Hamiltonian is now written as follows:

$$\begin{aligned} H = &\sum_{ij}\left[\frac{\omega_{pj}^2}{2}\left(P_{ij}^{(T)} - \sum_k q_k \int_{V_m} \boldsymbol{\alpha}_k^{(T)} \cdot \boldsymbol{\beta}_i^{(T)} dv\right)^2 + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(T)2}\right] \\ &+\sum_{ij}\left[\frac{\omega_{pj}^2}{2}\left(P_{ij}^{(s)} - \sum_k q_k \int_{V_m} \boldsymbol{\alpha}_k^{(T)} \cdot \boldsymbol{\beta}_i^{(s)} dv\right)^2 + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(s)2}\right] \\ &+\frac{1}{2}\sum_{ik} K_{ik}\left(\sum_j X_{ij}^{(s)}\right)\left(\sum_{j'} X_{kj'}^{(s)}\right) \\ &+\sum_i\left(\frac{1}{2}p_i^2 + \frac{1}{2}\Omega_i^2 q_i^2\right) \\ &+\sum_{ij}\left(\frac{\omega_{pj}^2}{2} P_{ij}^{(b)2} + \frac{\omega_{0j}^2}{2\omega_{pj}^2} X_{ij}^{(b)2}\right) + \frac{1}{2}\sum_i\left(\sum_j X_{ij}^{(b)}\right)^2, \end{aligned} \tag{28}$$

where

$$K_{ik} \equiv \int_{V_m} \boldsymbol{\alpha}_i^{(s)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) dv = \int_{V_m} \boldsymbol{\beta}_i^{(s)}(\mathbf{r}) \cdot \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) dv. \tag{29}$$

For simplicity, we introduce new conjugate variables $\tilde{q}_i$ and $\tilde{p}_i$ by canonical transformation $p_i = -\tilde{q}_i$ and $q_i = \tilde{p}_i$. If we define column vectors $\mathbf{x} =^t (\{X_{ij}^{(T)}\}, \{X_{ij}^{(s)}\}, \{\tilde{q}_i\}, \{X_{ij}^{(b)}\})$ and $\mathbf{p} =^t (\{P_{ij}^{(T)}\}, \{P_{ij}^{(s)}\}, \{\tilde{p}_i\}, \{P_{ij}^{(b)}\})$, the total Hamiltonian can be written in a compact form as follows:

$$H = \frac{1}{2}{}^t\mathbf{p} T \mathbf{p} + \frac{1}{2}{}^t\mathbf{x} V \mathbf{x}, \tag{30}$$

where T and V are symmetric matrices. Since T is positive definite, it has a decomposition $T = A^2$ by a symmetric regular matrix A. The positive semidefinite Hermite matrix AVA can be diagonalized by an orthogonal matrix U as ${}^tUAVAU = D$, where D is a diagonal matrix with no negative elements. A transformation to new bases $\mathbf{X} = {}^t(X_1, X_2, \ldots)$ and $\mathbf{P} = {}^t(P_1, P_2, \ldots)$ defined as $\mathbf{X} = {}^tUA^{-1}\mathbf{x}$ and $\mathbf{p} = A^{-1}U\mathbf{P}$ is a canonical transformation since the commutation relations are preserved:

$$
\begin{aligned}
[X_i, P_j] &= [\sum_k ({}^tUA^{-1})_{ik} x_k, \sum_l ({}^tUA)_{jl} p_k] = \sum_{kl} ({}^tUA^{-1})_{ik} ({}^tUA)_{jl} [x_k, p_l] \\
&= i\hbar \sum_k ({}^tUA^{-1})_{ik} (AU)_{kj} = i\hbar\delta_{ij}.
\end{aligned} \tag{31}
$$

This transformation diagonalizes the Hamiltonian (30) into a sum of independent harmonic oscillators:

$$
H = \frac{1}{2}{}^t\mathbf{P}\mathbf{P} + \frac{1}{2}{}^t\mathbf{X}D\mathbf{X} = \sum_i \frac{1}{2}(P_i^2 + \omega_i^2 X_i^2) + \sum_k \frac{1}{2}(P_k^{(b)2} + \omega_k^{(b)2} X_k^{(b)2}), \tag{32}
$$

where we have separated the contribution of bulk plasmons and suffixed their variables with superscript (b) since they are decoupled from the others.

Now we second quantize Eq. (32) to obtain

$$
H = \sum_i \hbar\omega_i (a_i^\dagger a_i + \frac{1}{2}) + \sum_k \hbar\omega_k^{(b)} (a_k^{(b)\dagger} a_k^{(b)} + \frac{1}{2}), \tag{33}
$$

where we introduced creation and annihilation operators in the usual way. The energy of the ground state $\langle H \rangle_0$ is

$$
\langle H \rangle_0 = E_0 + E_0^{(b)} \equiv \frac{1}{2}\sum_i \hbar\omega_i + \frac{1}{2}\sum_k \hbar\omega_k^{(b)}. \tag{34}
$$

The Casimir force is derived from the change in the ground-state energy with respect to an infinitesimal displacement of the dielectric slabs. As we have seen, the energy $E_0^{(b)}$ contributed by the bulk plasmons is independent of this displacement. Thus the Casimir force originates from E_0—the sum of zero-point energies of the harmonic-oscillator modes X_i. The relation $\mathbf{X} = {}^tUA^{-1}\mathbf{x}$ implies that X_i is a linear combination of $\tilde{q}_i$, $X_{ij}^{(T)}$, and $X_{ij}^{(s)}$. This means that X_i represents a combined mode of the EM field and the collective modes of charges in the dielectric. Therefore, in our approach, the Casimir force is attributable neither to the change in zero-point energies of the genuine EM field nor to that in zero-point energies of the genuine matter, but to that in *zero-point energies of the combined matter-field modes.*

III. EQUIVALENCE TO MAXWELL EQUATIONS

In this section, we show that the diagonalization of the Hamiltonian (28) amounts to solving the Maxwell equations associated with a proper dielectric response function $\epsilon(\omega)$ under appropriate boundary conditions.

Consider the equations of motion derived from the Hamiltonian (28). After eliminating q_i, $P_{ij}^{(T)}$, $P_{ij}^{(s)}$, and $P_{ij}^{(b)}$, these equations read

$$
\ddot{p}_i = -\Omega_i^2 p_+ \sum_{kj} \ddot{X}_{kj}^{(T)} \int_{V_m} \boldsymbol{\alpha}_i^{(T)} \cdot \boldsymbol{\beta}_k^{(T)} dv + \sum_{kj} \ddot{X}_{kj}^{(s)} \int_{V_m} \boldsymbol{\alpha}_i^{(T)} \cdot \boldsymbol{\beta}_k^{(s)} dv, \tag{35}
$$

$$
\ddot{X}_{kj}^{(T)} = -\omega_{0j}^2 X_{kj}^{(T)} - \omega_{pj}^2 \sum_i p_i \int_{V_m} \boldsymbol{\beta}_k^{(T)} \cdot \boldsymbol{\alpha}_i^{(T)} dv, \tag{36}
$$

$$
\ddot{X}_{kj}^{(s)} = -\omega_{0j}^2 X_{kj}^{(s)} - \omega_{pj}^2 \sum_i p_i \int_{V_m} \boldsymbol{\beta}_k^{(s)} \cdot \boldsymbol{\alpha}_i^{(T)} dv - \omega_{pj}^2 \sum_l K_{kl} \sum_{j'} X_{lj'}^{(s)}, \tag{37}
$$

$$
\ddot{X}_{kj}^{(b)} = -\omega_{0j}^2 X_{kj}^{(b)} - \omega_{pj}^2 \sum_{j'} X_{kj'}^{(b)}. \tag{38}
$$

The diagonalization of the Hamiltonian is equivalent to finding the solution to the above set of equations that oscillates at frequency ω. For such a solution, the second-order time derivative may be replaced by $-\omega^2$. Equation (35) then reduces to

$$p_i = \frac{\omega^2}{\omega^2 - \Omega_i^2} \left(\sum_k X_k^{(T)} \int_{V_m} \boldsymbol{\alpha}_i^{(T)} \cdot \boldsymbol{\beta}_k^{(T)} dv + \sum_k X_k^{(s)} \int_{V_m} \boldsymbol{\alpha}_i^{(T)} \cdot \boldsymbol{\beta}_k^{(s)} dv \right), \tag{39}$$

where we used the notation

$$X_k^{(M)} \equiv \sum_j X_{kj}^{(M)} \quad (M = T, s, b). \tag{40}$$

From the remaining equations (36)–(38), we obtain after summation over the suffix j

$$X_k^{(T)} = (1 - \epsilon(\omega)) \sum_i p_i \int_{V_m} \boldsymbol{\beta}_k^{(T)} \cdot \boldsymbol{\alpha}_i^{(T)} dv, \tag{41}$$

$$X_k^{(s)} = (1 - \epsilon(\omega)) \left[\sum_i p_i \int_{V_m} \boldsymbol{\beta}_k^{(s)} \cdot \boldsymbol{\alpha}_i^{(T)} dv + \sum_l X_l^{(s)} \int_{V_m} \boldsymbol{\beta}_k^{(s)} \cdot \boldsymbol{\alpha}_l^{(s)} dv \right], \tag{42}$$

$$X_k^{(b)} = (1 - \epsilon(\omega)) X_k^{(b)}, \tag{43}$$

where we have introduced the dielectric response function $\epsilon(\omega)$ defined as

$$\epsilon(\omega) \equiv 1 - \sum_j \frac{\omega_{pj}^2}{\omega^2 - \omega_{0j}^2}. \tag{44}$$

Let us introduce the electric field $\mathbf{E}(\mathbf{r})$ defined as

$$\mathbf{E}(\mathbf{r}) \equiv -\frac{1}{\sqrt{\epsilon_0}} \left(\sum_i p_i \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) + \sum_k X_k^{(s)} \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) + \sum_l X_l^{(b)} \boldsymbol{\alpha}_l^{(b)}(\mathbf{r}) \right). \tag{45}$$

Then it is easy to show that Eqs. (41)–(43) have the same form as follows:

$$X_k^{(M)} = \sqrt{\epsilon_0} (\epsilon(\omega) - 1) \int_{V_m} \boldsymbol{\beta}_k^{(M)} \cdot \mathbf{E} dv \qquad (M = T, s, b). \tag{46}$$

Substituting this into Eq. (39) and noting the completeness relation

$$\sum_i \boldsymbol{\beta}_i^{(T)}(\mathbf{r}) \boldsymbol{\beta}_i^{(T)}(\mathbf{r}') + \sum_k \boldsymbol{\beta}_k^{(s)}(\mathbf{r}) \boldsymbol{\beta}_k^{(s)}(\mathbf{r}') + \sum_l \boldsymbol{\beta}_l^{(b)}(\mathbf{r}) \boldsymbol{\beta}_l^{(b)}(\mathbf{r}') = \overleftrightarrow{1} \delta^3(\mathbf{r} - \mathbf{r}'), \tag{47}$$

where $\overleftrightarrow{1}$ denotes the identity matrix, we obtain

$$p_i = \frac{\sqrt{\epsilon_0} (\epsilon(\omega) - 1) \omega^2}{\omega^2 - \Omega_i^2} \int_{V_m} \boldsymbol{\alpha}_i^{(T)} \cdot \mathbf{E} dv. \tag{48}$$

Substituting Eqs. (46) and (48) into Eq. (45) yields the self-consistent condition for the electric field $\mathbf{E}(\mathbf{r})$:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) = [1 - \epsilon(\omega)] \Bigg(&\sum_i \frac{\omega^2}{\omega^2 - \Omega_i^2} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \int_{V_m} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dv' \\ &+ \sum_k \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_k^{(s)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dv' + \sum_l \boldsymbol{\alpha}_l^{(b)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dv' \Bigg). \end{aligned} \tag{49}$$

This equation can be simplified if we convert it into a pair of equations as follows. First, by taking the divergence of both sides we obtain

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{E}(\mathbf{r}) = [1 - \epsilon(\omega)] \Bigg(&\sum_k \boldsymbol{\nabla} \cdot \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_k^{(s)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dv' \\ &+ \sum_l \boldsymbol{\nabla} \cdot \boldsymbol{\alpha}_l^{(b)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}') dv' \Bigg). \end{aligned} \tag{50}$$

Substituting Eq. (16) into Eq. (50), we obtain

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = [1-\epsilon(\omega)]\nabla \cdot \left(\sum_k \boldsymbol{\beta}_k^{(s)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_k^{(s)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dv' \right.$$
$$\left. + \sum_l \boldsymbol{\beta}_l^{(b)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dv' \right) \Theta(\mathbf{r}), \tag{51}$$

where $\Theta(\mathbf{r})$ is the unit step function defined by

$$\Theta(\mathbf{r}) = \begin{cases} 1 & \text{in } V_m, \\ 0 & \text{otherwise.} \end{cases} \tag{52}$$

Using the completeness relation (47) and noting that $\boldsymbol{\beta}_i^{(T)}(\mathbf{r})\Theta(\mathbf{r})$ is a transverse function in V, Eq. (51) is simplified, giving

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = [1-\epsilon(\omega)]\nabla \cdot [\mathbf{E}(\mathbf{r})\Theta(\mathbf{r})]. \tag{53}$$

On the other hand, by operating ($\omega^2 - c^2\text{rotrot}$) on both sides of Eq. (49) and using Eq. (10), we have

$$(\omega^2 - c^2\text{rotrot})\mathbf{E}(\mathbf{r}) = [1-\epsilon(\omega)]\omega^2 \left(\sum_i \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \int_{V_m} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dv' \right.$$
$$\left. + \sum_k \boldsymbol{\alpha}_k^{(s)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_k^{(s)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dv' + \sum_l \boldsymbol{\alpha}_l^{(b)}(\mathbf{r}) \int_{V_m} \boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \cdot \mathbf{E}(\mathbf{r}')dv' \right). \tag{54}$$

Here, using (17) and (47), we obtain

$$\begin{aligned}
&\sum_i \boldsymbol{\alpha}_i^{(T)}(\mathbf{r})\boldsymbol{\alpha}_i^{(T)}(\mathbf{r}') + \sum_k \boldsymbol{\alpha}_k^{(s)}(\mathbf{r})\boldsymbol{\beta}_k^{(s)}(\mathbf{r}') + \sum_l \boldsymbol{\alpha}_l^{(b)}(\mathbf{r})\boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \\
&= \sum_k \Theta(\mathbf{r})\boldsymbol{\beta}_k^{(s)}(\mathbf{r})\boldsymbol{\beta}_k^{(s)}(\mathbf{r}') + \sum_l \Theta(\mathbf{r})\boldsymbol{\beta}_l^{(b)}(\mathbf{r})\boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \\
&\quad + \sum_i \boldsymbol{\alpha}_i^{(T)}(\mathbf{r})\boldsymbol{\alpha}_i^{(T)}(\mathbf{r}') - \sum_{ik} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \int_{V_m} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}'') \cdot \boldsymbol{\beta}_k^{(s)}(\mathbf{r}'')dv''\boldsymbol{\beta}_k^{(s)}(\mathbf{r}') \\
&= \sum_k \Theta(\mathbf{r})\boldsymbol{\beta}_k^{(s)}(\mathbf{r})\boldsymbol{\beta}_k^{(s)}(\mathbf{r}') + \sum_l \Theta(\mathbf{r})\boldsymbol{\beta}_l^{(b)}(\mathbf{r})\boldsymbol{\beta}_l^{(b)}(\mathbf{r}') \\
&\quad + \sum_{ik} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}) \int_{V_m} \boldsymbol{\alpha}_i^{(T)}(\mathbf{r}'') \cdot \boldsymbol{\beta}_k^{(T)}(\mathbf{r}'')dv''\boldsymbol{\beta}_k^{(T)}(\mathbf{r}') \\
&= \Theta(\mathbf{r})\overleftrightarrow{1}\delta^3(\mathbf{r}-\mathbf{r}'),
\end{aligned} \tag{55}$$

where we have used the fact that $\Theta(\mathbf{r}'')\boldsymbol{\beta}_k^{(T)}(\mathbf{r}'')$ can be expanded with $\boldsymbol{\alpha}_i^{(T)}(\mathbf{r}'')$. We thus obtain

$$(\omega^2 - c^2\text{rotrot})\mathbf{E}(\mathbf{r}) = [1-\epsilon(\omega)]\omega^2\mathbf{E}(\mathbf{r})\Theta(\mathbf{r}). \tag{56}$$

If we define the electric displacement $\mathbf{D}(\mathbf{r})$ as $\mathbf{D}(\mathbf{r}) \equiv \epsilon_0\{1 + [\epsilon(\omega) - 1]\Theta(\mathbf{r})\}\mathbf{E}(\mathbf{r})$, we find that Eqs. (53) and (56) are identical to the Maxwell equations, namely,

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = 0 \tag{57}$$

and

$$\nabla \times (\nabla \times \mathbf{E}(\mathbf{r})) = \frac{\omega^2}{\epsilon_0 c^2}\mathbf{D}(\mathbf{r}). \tag{58}$$

Our theory developed in Sec. II therefore yields the same sequence of eigenvalues ω_i as Schram's method [4] and eigenvalues of the bulk plasmon modes that are not relevant to the Casimir force. Our theory thus gives the identical

result for the Casimir force: for example, when the interface S_m consists of a pair of parallel planes with a distance d, the force per area is calculated to be

$$F = -\frac{\hbar}{2\pi^2 c^3}\int_0^\infty d\xi \int_1^\infty dp p^2 \xi^3 \left[\left\{ \left(\frac{s + p\epsilon(i\xi)}{s - p\epsilon(i\xi)} \right)^2 e^{2p\xi d/c} - 1 \right\}^{-1} + \left\{ \left(\frac{s+p}{s-p} \right)^2 e^{2p\xi d/c} - 1 \right\}^{-1} \right], \tag{59}$$

where $s \equiv \sqrt{p^2 + \epsilon i\xi - 1}$.

IV. FIELD THEORIES AND MATTER THEORIES OF THE CASIMIR FORCE

In the preceding sections, we have developed a theory of the Casimir force by starting from the Lagrangian that describes both the EM field and the collective modes of charges in the matter. In this theory, the Casimir effect is attributed to a change in zero-point energies of the combined modes of the EM field and the matter. Thus, in our theory, both field and matter contribute to the Casimir force. The system considered here, however, has one asymmetry between the field and matter, which is obvious from Fig. 1. The asymmetry originates from the property of the interface S_m in Fig. 1. It confines the matter in one side, but imposes no boundary condition on the EM field. It is difficult to imagine an opposite situation. That is, we can exclude the matter from some region, but we cannot clear away the EM vacuum. As will be discussed below, it is this asymmetry that makes some of the existing theories favor the interpretation of the Casimir force as arising from zero-point EM energies (field theories), and others favor the interpretation as due to zero-point fluctuations of the matter (matter theories).

First, we consider a field theory of the Casimir force, by which we refer to the scheme of solving the Maxwell equations under appropriate boundary conditions. The relation between our theory and the field theory has already been discussed in Sec. III. There we have shown that both theories actually calculate the zero-point energies of the same set of combined matter-field modes. Nevertheless these energies are sometimes incorrectly identified as zero-point energies of the genuine EM field, presumably because the normal modes can be specified by only invoking the Maxwell equations without any reference to the state of the matter. The degrees of freedom of the matter are embedded in the dielectric response function from the very beginning in the field theory of the Casimir force and are therefore not manifest. In addition, in the simplest case of the two perfectly conducting plates, it would be easy to forget the matter because it occupies only the thin surface and the whole mode volume is filled with the pure EM field. As a matter of fact, however, within this thin surface region there exist rather complicated surface modes, as described in Sec. II, which conspire to shield the EM field, producing the Casimir effect. In contrast, these normal modes can hardly be looked upon as genuine matter oscillations because it is difficult to construct a simple wave equation that contains only the matter variables when the system includes regions where no matter exists.

A matter theory of the Casimir force, which is based on zero-point fluctuations of the matter, is formulated by Milonni and Shih [21] in terms of conventional quantum electrodynamics. Their scheme looks somewhat complicated, but the essential ingredient of the theory is the use of second-order perturbation theory in calculating the self-energy of linearly interacting harmonic oscillators. Here we will present the bare essentials of this theory by taking up the simplest example of a pair of interacting harmonic oscillators.

The Hamiltonian for the system is written as

$$H = H_0 + V, \tag{60}$$

where

$$H_0 = \hbar\omega_a \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + \hbar\omega_b \left(\hat{b}^\dagger \hat{b} + \frac{1}{2} \right), \tag{61}$$

$$V = v_+ \hat{a}^\dagger \hat{b}^\dagger + v_- \hat{a}^\dagger \hat{b} + v_+^* \hat{b}\hat{a} + v_-^* \hat{b}^\dagger \hat{a}, \tag{62}$$

and the interaction V is assumed to be small. In this case, the first nonvanishing correction to the ground-state energy is of second order in V and is written as $\Delta E^{(2)} = \langle g^{(0)}|V|g\rangle$, where the state $|g^{(0)}\rangle$ is the unperturbed ground state and $|g\rangle$ is the ground state, which is correct up to first order in V. Instead of using a time-independent perturbation theory, we can also obtain $|g\rangle$ by a time-dependent perturbation theory with the Hamiltonian $H_0 + Ve^{\gamma t}$, where the limit $\gamma \to +0$ is implied. Hence

$$\begin{aligned} |g\rangle &= \left(1 - \frac{i}{\hbar}\int_{-\infty}^0 dt' e^{iH_0 t'} V e^{\gamma t'} e^{-iH_0 t'} \right) |g^{(0)}\rangle \\ &\equiv U|g^{(0)}\rangle. \end{aligned} \tag{63}$$

Up to second order, we have

$$\begin{aligned}\Delta E^{(2)} &= \langle g^{(0)}|VU|g^{(0)}\rangle = \langle g^{(0)}|U^\dagger V|g^{(0)}\rangle \\ &= \frac{1}{2}\langle g^{(0)}|U^\dagger VU|g^{(0)}\rangle. \end{aligned} \tag{64}$$

This corresponds to the interaction energy assumed in Ref. [21]. The following relations are useful for the following calculation:

$$U^\dagger \hat{a} U = \hat{a} - \frac{v_+}{\omega_a+\omega_b}\hat{b}^\dagger - \frac{v_-}{\omega_a-\omega_b}\hat{b}, \tag{65}$$

$$U^\dagger \hat{b} U = \hat{b} - \frac{v_+}{\omega_a+\omega_b}\hat{a}^\dagger - \frac{v_-}{\omega_b-\omega_a}\hat{a}. \tag{66}$$

Since the interaction V consists of the products of commuting operators, changing the ordering of operators in V should not affect the value of $\Delta E^{(2)}$. Let us take one particular ordering of

$$V = v_+\hat{a}^\dagger\hat{b}^\dagger + v_-\hat{a}^\dagger\hat{b} + v_+^*\hat{b}\hat{a} + v_-^*\hat{b}^\dagger\hat{a}, \tag{67}$$

and substitute Eqs. (65) and (66) directly into $U^\dagger VU$ without changing the order of operators. In the resulting expression, the operators for mode a appear in normal order, and therefore do not contribute to the expectation value in the vacuum. Thus, $\Delta E^{(2)}$ in Eq. (64) is expressed only in terms of zero-point fluctuations of mode b as

$$\Delta E^{(2)} = -\frac{|v_+|^2}{\omega_a+\omega_b}\langle g^{(0)}|\hat{b}\hat{b}^\dagger|g^{(0)}\rangle. \tag{68}$$

If we take another choice of operator ordering for V,

$$V = v_+\hat{b}^\dagger\hat{a}^\dagger + v_-\hat{a}^\dagger\hat{b} + v_+^*\hat{a}\hat{b} + v_-^*\hat{b}^\dagger\hat{a}, \tag{69}$$

the same procedure leads to

$$\Delta E^{(2)} = -\frac{|v_+|^2}{\omega_a+\omega_b}\langle g^{(0)}|\hat{a}\hat{a}^\dagger|g^{(0)}\rangle. \tag{70}$$

In this case, the correction is attributable to quantum fluctuations of mode a. The second-order correction to the ground-state energy of a pair of linearly coupled harmonic oscillators can therefore be attributed solely to zero-point fluctuations of either one of the oscillators.

Now, back to the original problem of the Casimir force. We notice [21] that the force is derived from the change in the energy of the entire system when we add atoms in an infinitesimally thin layer next to the boundary S_m [22]. The dominant contribution to this change arises from the interaction energy between the added atoms and the field modes, either of which may be described as a set of harmonic oscillators. Thus, as in the simple system discussed above, we can attribute the change in energy to the fluctuations of the added atoms alone, provided that an appropriate operator ordering is chosen (matter theory). We emphasize that the logical consistency of this interpretation hinges heavily on second-order perturbation theory as applied to a system of linearly coupled harmonic oscillators (linear-response theory). If effects of higher-order interactions are not negligible, it would be nontrivial to construct either genuine matter theory or field theory that is consistent with each other.

If we choose another appropriate operator ordering, the Casimir force may be interpreted as arising from quantum fluctuations of the EM field modes, and we obtain a kind of field theory of the Casimir force. But the machinery is similar to the field theories described earlier, that is, the field modes are actually not the genuine EM field modes but the combined modes of the EM field and the matter.

Second-order perturbation theory (or linear-response theory) thus allows the interpretation of the Casimir force as arising from quantum fluctuations of the genuine matter alone but not from those of the genuine EM field alone. The reason for this asymmetry lies again in the property of the boundary. That is, an infinitesimal displacement of the boundary only affects the mode functions of the genuine matter but has no direct effect on the mode functions of the genuine EM field.

V. SUMMARY

The origin of the Casimir force has usually been attributed to zero-point fluctuations of the EM field, and sometimes to those of the matter. We have formulated the problem by starting from the Lagrangian that describes the EM field, collective modes of the matter, and their interaction. This approach makes it clear that the Casimir effect, in actual fact, arises from the change in zero-point energies of certain combined modes of the transverse EM field, the transverse motion of charges, and the surface plasmons. The zero-point energy of the entire system is therefore contributed by both the EM field and the matter.

The Casimir force arises not from the zero-point energy *per se*, but from its change with respect to a virtual infinitesimal displacement of the plates. This change alters the mode functions of the matter alone, and does not affect those of the EM field. Because of this asymmetry, it is possible to construct a genuine matter theory but it is difficult to conceive a genuine field theory.

A most natural derivation of the Casimir force seems to be to calculate the zero-point energy first and then take its derivative. The original derivation by Casimir follows this approach. In this type of approach, both field and matter contribute to the zero-point energy as in our theory. Nevertheless, one encounters descriptions in the literature to the effect that the Casimir force is the hallmark of zero-point energies of the genuine EM field. Perhaps the main reason for this (strictly speaking incorrect) recognition lies in the fact that while the system has the region where only the EM field (including the vacuum) exists, there is no region where only the matter exists. This also reflects the above-mentioned asymmetry between the field and the matter.

ACKNOWLEDGMENT

M.U. acknowledges support by the Core Research for Evolutional Science and Technology (CREST) of the Japan Science and Technology Corporation.

[1] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. **51**, 793 (1948).
[2] E. M. Lifshitz, Zh. Eksp. Teor. Fiz. **29**, 94 (1955) [Sov. Phys. JETP **2**, 73 (1956)].
[3] N. G. van Kampen, B. R. A. Nijboer, and K. Schram, Phys. Lett. **26A**, 307 (1968).
[4] K. Schram, Phys. Lett. **43A**, 282 (1973).
[5] T. H. Boyer, Phys. Rev. **174**, 1764 (1968).
[6] M. J. Sparnaay, Physica (Utrecht) **24**, 751 (1958).
[7] D. Tabor and R. H. S. Winterton, Nature (London) **219**, 1120 (1968).
[8] J. N. Israelachvili and D. Tabor, Proc. R. Soc. London A **331**, 19 (1972).
[9] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).
[10] H. B. G. Casimir and D. Polder, Phys. Rev. **73**, 360 (1948).
[11] C. I. Sukenik, M. G. Boshier, D. Cho, V. Sandoghdar, and E. A. Hinds, Phys. Rev. Lett. **70**, 560 (1993).
[12] J. Mehra, Physica (Utrecht) **37**, 145 (1967).
[13] L. S. Brown and G. J. Maclay, Phys. Rev. **184**, 1272 (1969).
[14] J. Schwinger, L. L. DeRaad, and K. A. Milton, Ann. Phys. (N.Y.) **115**, 1 (1978).
[15] X. Kong and F. Ravndal, Phys. Rev. Lett. **79**, 545 (1997).
[16] G. Plunien, B. Müller, and W. Greiner, Phys. Rep. **134**, 87 (1986).
[17] V. M. Mostepanenko and N. N. Trunov, Usp. Fiz. Nauk. **156**, 385 (1988) [Sov. Phys. Usp. **31**, 965 (1988)].
[18] E. Elizalde and A. Romeo, Am. J. Phys. **59**, 711 (1991).
[19] L. Spruch, Science **272**, 1452 (1996).
[20] J. Schwinger, Lett. Math. Phys. **1**, 43 (1975).
[21] P. W. Milonni and M. -L. Shih, Phys. Rev. A **45**, 4241 (1992).
[22] Strictly speaking, when we add some amounts of atoms on one surface of the dielectric slab, we should remove the same amount of atoms from the other surface so that the volume of the slab is kept constant; otherwise the bulk plasmons would also contribute to the change in energy. We should take into account both effects of adding and removing atoms in evaluating the change in energy.